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Solutions of the Einstein–Dirac equation on Riemannian 3-manifolds with constant scalar curvature☆

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Abstract

This paper contains a classification of all three-dimensional manifolds with constant eigenvalues of the Ricci tensor that carry a non-trivial solution of the Einstein–Dirac equation.

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1. Introduction

Consider a Riemannian spin manifold of dimension $n \geq 3$ and denote by D the Dirac operator acting on spinor fields. A solution of the Einstein–Dirac equation is a spinor field ψ solving the equations

$$\mathrm{Ric} - \tfrac{1}{2} S \cdot g = \pm \tfrac{1}{4} T_\psi, \qquad D(\psi) = \lambda \psi.$$

Here S denotes the scalar curvature of the space, λ is a real constant and T_ψ the energy–momentum tensor of the spinor field ψ defined by the formula

$$T_\psi(X, Y) = (X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, \psi).$$

The scalar curvature S is related to the eigenvalue λ and the length of the spinor field ψ by the formula

$$S = \pm \frac{\lambda}{n-2} |\psi|^2.$$

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In [3] we introduced the weak Killing equation for a spinor field ψ^*:

$$\nabla_X \psi^* = \frac{n}{2(n-1)}\,\mathrm{d}S(X)\psi^* + \frac{2\lambda}{(n-2)S}\mathrm{Ric}(X)\cdot\psi^* - \frac{\lambda}{n-2}X\cdot\psi^* + \frac{1}{2(n-1)S}X\cdot \mathrm{d}S\cdot\psi^*$$

Any weak Killing spinor ψ^* (WK-spinor) yields a solution ψ of the Einstein–Dirac equation after normalization

$$\psi = \sqrt{\frac{(n-2)|S|}{|\lambda||\psi^*|^2}}\,\psi^*.$$

In fact, in dimension $n = 3$ the Einstein–Dirac equation is essentially equivalent to the weak Killing equation (see [2,3]). Up to now the following three-dimensional Riemannian manifolds admitting WK-spinors are known:

1. the flat torus T^3 with a parallel spinor;
2. the sphere S^3 with a Killing spinor;
3. two non-Einstein Sasakian metrics on the sphere S^3 admitting WK-spinors. The scalar curvature of these two left-invariant metrics equals $S = 1 \pm \sqrt{5}$.

The aim of this paper is to classify all Riemannian 3-manifolds with constant eigenvalues of the Ricci tensor and admitting a solution of the Einstein–Dirac equation. In particular, we will prove the existence of a one-parameter family of left-invariant metrics on S^3 with WK-spinors. This family contains the two non-Einstein Sasakian metrics with WK-spinors on S^3, but does not contain the standard sphere S^3 with Killing spinors. Moreover, any simply-connected, complete Riemannian manifold $N^3 \neq S^3$ with WK-spinors and constant scalar curvature is isometric to a space of this one-parameter family. In order to formulate the result precisely, we fix real parameters $K, L, M \in \mathbb{R}$ and denote by $N^3(K, L, M)$ the three-dimensional, simply-connected and oriented Riemannian manifold defined by the following structure equations:

$$\omega_{12} = K\sigma^3, \qquad \omega_{13} = L\sigma^2, \qquad \omega_{23} = M\sigma^1,$$

or, equivalently:

$$\mathrm{d}\sigma^1 = (L-K)\sigma^2\wedge\sigma^3, \qquad \mathrm{d}\sigma^2 = (M+K)\sigma^1\wedge\sigma^3, \qquad \mathrm{d}\sigma^3 = (L-M)\sigma^1\wedge\sigma^2.$$

The one-forms $\sigma^1, \sigma^2, \sigma^3$ are the dual forms of an orthonormal frame of vector fields. Using this frame the Ricci tensor of $N^3(K, L, M)$ is given by the matrix

$$\mathrm{Ric} = \begin{pmatrix} -2KL & 0 & 0 \\ 0 & 2KM & 0 \\ 0 & 0 & -2LM \end{pmatrix}.$$

Main Theorem. *Let $N^3 \neq S^3$ be a complete, simply-connected Riemannian manifold with constant eigenvalues of the Ricci tensor and scalar curvature $S \neq 0$. If N^3 admits a*

WK-spinor, then N^3 *is isometric to* $N^3(K, L, M)$ *and the parameters are a solution of the equation*

$$-K^2L(L-M)^2M + L^3M^3 + KL^2M^2(M-L) + K^3(L-M)(L+M)^2 = 0 \quad (*)$$

Conversely, any space $N^3(K, L, M)$ *such that* $(K, L, M) \neq (0, 0, 0)$ *is a solution of* (*) *admits two WK-spinors for one and only one WK-number* λ. *With respect to the fixed orientation of* $N^3(K, L, M)$ *we have the two cases:*

$$\lambda = \begin{cases} \dfrac{S}{2\sqrt{2}}\sqrt{\dfrac{S}{S^2 - 2|\mathrm{Ric}|^2}} & \text{if} \quad -K < M, \\ \lambda = -\dfrac{S}{2\sqrt{2}}\sqrt{\dfrac{S}{S^2 - 2|\mathrm{Ric}|^2}} & \text{if} \quad M < -K. \end{cases}$$

The spaces $N^3(K, L, M)$ *are isometric to* S^3 *equipped with a left-invariant metric*.

Remark. *If the parameters* $K = M$ *coincide, the solution of Eq.* (*) *is given by*

$$L = \tfrac{1}{4}K(1-\sqrt{5}), \qquad L = \tfrac{1}{4}K(1+\sqrt{5})$$

and we obtain the Ricci tensors

$$\mathrm{Ric} = \begin{pmatrix} \dfrac{1}{2}K^2(\sqrt{5}-1) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & \dfrac{1}{2}K^2(\sqrt{5}-1) \end{pmatrix}$$

or

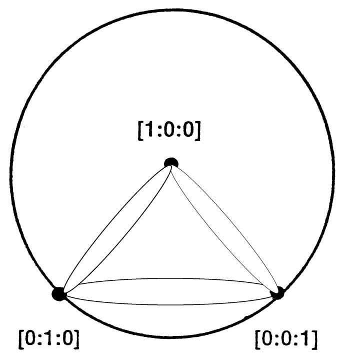
$$\mathrm{Ric} = \begin{pmatrix} -\dfrac{1}{2}K^2(1+\sqrt{5}) & 0 & 0 \\ 0 & 2K^2 & 0 \\ 0 & 0 & -\dfrac{1}{2}K^2(1+\sqrt{5}) \end{pmatrix}.$$

The non-Einstein–Sasakian metrics on S^3 *occur for the parameter* $K = 1$ (*see* [3]).

Remark. *Using the standard basis of the Lie algebra* $\mathfrak{so}(3)$ *we can write the left-invariant metric of the space* $N^3(K, L, M)$ *in the following way:*

$$\begin{pmatrix} \dfrac{1}{|M-L||K+M|} & 0 & 0 \\ 0 & \dfrac{1}{|K-L||M-L|} & 0 \\ 0 & 0 & \dfrac{1}{|K-L||K+M|} \end{pmatrix}.$$

Eq. (*) is a homogeneous equation of order six. The transformation $(K, L, M) \to (\mu K, \mu L, \mu M)$ *corresponds to a homothety of the metric. Therefore, up to a homothety, the moduli space of solutions is a subset of the real projective space* $\mathbb{P}^2(\mathbb{R})$ *given by Eq. (*). This subset is a configuration of six curves in* $\mathbb{P}^2(\mathbb{R})$ *connecting the three points* $[K : L : M]$=[1:0:0], [0:1:0], [0:0:1] *corresponding to flat metrics.*



In particular, we have constructed two paths of solutions of the Einstein–Dirac equation deforming the non-Einstein Sasakian metrics on S^3.

2. The integrability condition for the Einstein–Dirac equation in dimension $n = 3$

The spinor bundle of a three-dimensional Riemannian manifold is a complex vector bundle of dimension two. Moreover, there exists a quaternionic structure commuting with the Clifford multiplication by real vectors (see [1]). Consequently, in case of a real WK-number λ, the corresponding space of WK-spinors is a quaternionic vector space. In the spinor bundle let us introduce the metric connection ∇^λ given by the formula

$$\nabla^\lambda_X \psi := \nabla_X \psi - \frac{3}{4}\,\mathrm{d}S(X)\psi - \lambda \left\{ \frac{2}{S}\mathrm{Ric}(X) - X \right\} \cdot \psi - \frac{1}{4S} X \cdot \mathrm{d}S \cdot \psi$$

and denote by Ω^λ its curvature form. Then we obtain the following.

Proposition 1. *Let* N^3 *be a simply-connected three-dimensional Riemannian manifold and suppose that the scalar curvature* $S \neq 0$ *does not vanish. Then the following conditions are equivalent:*

1. N^3 *is a non-trivial solution of the Einstein–Dirac equation with real eigenvalue* λ;
2. N^3 *admits a WK-spinor with real WK-number* λ;
3. N^3 *admits two WK-spinors with real WK-number* λ;
4. *The curvature* $\Omega^\lambda \equiv 0$ *vanishes identically*.

If the scalar curvature $S \neq 0$ is constant, the condition $\Omega^\lambda \equiv 0$ has been investigated and yields algebraic equations involving the Ricci tensor and its covariant derivative (see [3], Theorem 8.3). In order to formulate the integrability condition, we denote by $X \times Y$

the vector cross product of two vectors $X, Y \in T(N^3)$. For brevity, let us introduce the endomorphism $T : T(N^3) \to T(N^3)$ given by the formula

$$T(X) = \sum_{i=1}^{3} e_i \times (\nabla_{e_i}\mathrm{Ric})(X),$$

which will be used in the proof of the Main Theorem.

Theorem 1 (See [3]). *Let N^3 be a simply-connected three-dimensional Riemannian manifold with constant scalar curvature $S \neq 0$. N^3 admits a solution of the Einstein–Dirac equation with real eigenvalue λ if and only if the following three conditions are satisfied:*

1. $8\lambda^2\{S^2 - 2|\mathrm{Ric}|^2\} = S^3$;
2. $8\lambda^2\{S\mathrm{Ric}(X) - 2\mathrm{Ric} \circ \mathrm{Ric}(X)\} - 4\lambda ST(X) - S^2\mathrm{Ric}(X) = 0$;
3. $8\lambda^2\{2\mathrm{Ric}(X) - SX\} \times \{2\mathrm{Ric}(Y) - SY\} + 8\lambda S\{(\nabla_X\mathrm{Ric})(Y) - (\nabla_Y\mathrm{Ric})(X)\}$
$+S^3 X \times Y = 2S^2\sum_{i<j}\{R_{jY}\delta_{iX} + R_{iX}\delta_{jY}\}e_i \times e_j.$

3. Proof of the Main Theorem

We fix an orthonormal frame e_1, e_2, e_3 of vector fields on N^3 consisting of eigenvectors of the Ricci tensor:

$$\mathrm{Ric} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}.$$

Denote by $\sigma^1, \sigma^2, \sigma^3$ the dual frame and consider the connection forms $\omega_{ij} = \langle \nabla e_i, e_j \rangle$ of the Levi–Civita connection. The structure equations of the Riemannian manifold N^3 are

$$\mathrm{d}\omega_{12} = \omega_{13} \wedge \omega_{32} + \tfrac{1}{2}(C - A - B)\sigma^1 \wedge \sigma^2,$$
$$\mathrm{d}\omega_{13} = \omega_{12} \wedge \omega_{23} + \tfrac{1}{2}(B - A - C)\sigma^1 \wedge \sigma^3,$$
$$\mathrm{d}\omega_{23} = \omega_{21} \wedge \omega_{13} + \tfrac{1}{2}(A - B - C)\sigma^2 \wedge \sigma^3,$$

and the covariant derivative $\nabla\mathrm{Ric}$ is given by the matrix of 1-forms

$$\nabla\mathrm{Ric} = \begin{pmatrix} \mathrm{d}A & (A-B)\omega_{12} & (A-C)\omega_{13} \\ (A-B)\omega_{12} & \mathrm{d}B & (B-C)\omega_{23} \\ (A-C)\omega_{13} & (B-C)\omega_{23} & \mathrm{d}C \end{pmatrix}.$$

The second equation of Theorem 1 yields the condition that all elements outside the diagonal of the (1,1)-tensor T are zero:

$$(A - B)\omega_{12}(e_1) = 0 = (A - B)\omega_{12}(e_2),$$

$$(C-A)\omega_{13}(e_1)=0=(C-A)\omega_{13}(e_3),$$

$$(B-C)\omega_{23}(e_2)=0=(B-C)\omega_{23}(e_3).$$

First, we discuss the generic case that A, B, C are pairwise different. Then there exist numbers K, L, M such that

$$\omega_{12}=K\sigma^3, \qquad \omega_{13}=L\sigma^2, \qquad \omega_{23}=M\sigma^1.$$

The parameter triples $\{A, B, C\}$ and $\{K, L, M\}$ are related via the structure equations by the formulas

$$A=-2KL, \qquad B=2KM, \qquad C=-2LM.$$

The first and second equation of Theorem 1 become equivalent to the following system of algebraic equations:

$$1'.\quad \lambda=\pm\frac{S}{2\sqrt{2}}\sqrt{\frac{S}{S^2-2|\mathrm{Ric}|^2}};$$

$$2'.\quad 2S(S^2-2|\mathrm{Ric}|^2)\{(A-C)L+(B-A)K\}^2=S(SA-2A^2)-A(S^2-2|\mathrm{Ric}|^2),$$

$$2S(S^2-2|\mathrm{Ric}|^2)\{(C-B)M+(A-B)K\}^2=S(SB-2B^2)-B(S^2-2|\mathrm{Ric}|^2),$$

$$2S(S^2-2|\mathrm{Ric}|^2)\{(B-C)M+(C-A)L\}^2=S(SC-2C^2)-C(S^2-2|\mathrm{Ric}|^2).$$

We solve this system of algebraic equations with respect to the parameters K, L, M. It turns out that $2'$ can be written in the form

$$P_i(K,L,M)\cdot Q(K,L,M)=0,$$

$(1\le i\le 3)$, where the polynomials P_1, P_2, P_3 and Q are given by the formulas

$$P_1(K,L,M)=(-KL^2+L^2M+K^2(L+M))^2,$$

$$P_2(K,L,M)=(KM^2+LM^2+K^2(L+M))^2,$$

$$P_3(K,L,M)=(LM(-L+M)+K(L^2+M^2))^2,$$

$$Q(K,L,M)=-K^2L(L-M)^2M+L^3M^3 +KL^2M^2(M-L)+K^3(L-M)(L+M)^2.$$

The real solutions of $P_1=P_2=P_3=0$ are the pairs $\{K=0, L=0\}$ (the flat metric) and $\{K=M, L=-M\}$ (the space of positive constant curvature). Therefore, we proved that a three-dimensional complete, simply-connected manifold N^3 with constant scalar curvature $S\neq 0$ and different eigenvalues of the Ricci tensor is isometric to one of the spaces $N^3(K, L, M)$, where the parameters K, L, M are solutions of the equation $Q(K, L, M)=0$. These spaces satisfy the conditions 1 and 2 of Theorem 1 and, moreover, a simple computation yields the result that condition 3 of Theorem 1 is satisfied, too. We next discuss the case that two of the eigenvalues A, B, C coincide, for example, $A = C \neq B$. Then we obtain again

$$\omega_{12} = K\sigma^3, \qquad \omega_{23} = M\sigma^1,$$

but there is no condition for the connection form ω_{13}. We compute the matrix of the (1, 1)-tensor T:

$$T = \begin{pmatrix} (B-C)K & 0 & 0 \\ 0 & (C-B)(K+M) & 0 \\ 0 & 0 & (B-C)M \end{pmatrix}.$$

Since the scalar curvature S as well as the eigenvalues $A = C, B$ of the Ricci tensor are constant, the second equation of Theorem 1 yields that K and M are constant and, moreover, coincide:

$$K = M = \text{constant}.$$

In case of $K = M = 0$ we have $\omega_{12} = \omega_{23} = 0$ and $A = C$. In particular, the Ricci tensor is parallel, $\nabla \mathrm{Ric} = 0$. Therefore, in this case N^3 is a Ricci-parallel three-dimensional manifold admitting a WK-spinor. Then N^3 is either flat or a space of constant positive curvature (see [3], Theorem 8.2). Finally, we consider the case of $K = M = 1$, i.e., $\omega_{12} = \sigma^3$ and $\omega_{23} = \sigma^1$. Differentiating the equation $\omega_{12} = \sigma^3$, we obtain

$$\omega_{13} \wedge \omega_{32} - \tfrac{1}{2} B\sigma^1 \wedge \sigma^2 = \mathrm{d}\omega_{12} = \mathrm{d}\omega^3 = \omega_{31} \wedge \sigma^1 + \omega_{32} \wedge \sigma^2$$
$$-\tfrac{1}{2} B\sigma^1 \wedge \sigma^2 = -\sigma^1 \wedge \sigma^2.$$

Consequently, $B = 2$ and the tensors T and Ric are given by the matrices

$$T = \begin{pmatrix} 2-C & 0 & 0 \\ 0 & 2(C-2) & 0 \\ 0 & 0 & 2-C \end{pmatrix}, \qquad \mathrm{Ric} = \begin{pmatrix} C & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & C \end{pmatrix}.$$

The second condition of Theorem 1 yields the equations ($S = 2 + 2C$):

$$8\lambda^2(SC - 2C^2) - 4\lambda S(2-C) - S^2C = 0,$$

$$8\lambda^2(2S - 8) + 8\lambda S(2-C) - 2S^2 = 0.$$

Solving these equations with respect to λ and C we obtain the three solutions:

1. $C = 2$ and $\lambda = \pm\frac{3}{2}$. Then N^3 is isometric to S^3.
2. $C = -1$ and $\lambda = 0$. Then the scalar curvature $S = 0$ is zero.
3. $C = \frac{1}{2}(-1 \pm \sqrt{5})$ and $\lambda = 1 \pm \frac{1}{2}\sqrt{5}$. These metrics are the non-Einstein Sasakian metrics on S^3 admitting WK-spinors (see [3]). The corresponding space is contained in the family $N^3(K, L, M)$.

We have discussed all possibilities and, therefore, we have finished the proof of the Main Theorem.

4. Moduli space of solutions

The moduli space of all three-dimensional Riemannian manifolds with constant scalar curvature $S \neq 0$ and WK-spinors is given by the triples $\{K, L, M\}$ of real numbers satisfying the equation of order six, $Q(K, L, M) = 0$. The polynomial Q is symmetric in $\{K, -L, M\}$. Denote by

$$\gamma_1 = K - L + M, \qquad \gamma_2 = -KL + KM - LM, \qquad \gamma_3 = -KLM$$

the elementary symmetric functions of these variables. Then we have

$$Q = 4\gamma_1\gamma_2\gamma_3 - \gamma_2^3 - 4\gamma_3^2.$$

Consider the projective variety $V_{\mathbb{C}} \subset \mathbb{P}^2(\mathbb{C})$ defined by the homogeneous polynomial Q:

$$V_{\mathbb{C}} = \{[K : L : M] \in \mathbb{P}^2(\mathbb{C}) : Q(K, L, M) = 0\}.$$

$V_{\mathbb{C}}$ has three singular points:

$$V_{\mathbb{C}}^{\mathrm{sing}} = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$$

and these points correspond to the flat metric. We will now parametrize the variety $V_{\mathbb{C}}$ by two meromorphic functions defined on a smooth Riemann surface. $V_{\mathbb{C}}$ is given by the equation ($K = 1$):

$$Q(1, L, M) = L^3(M-1)^2(M+1) + L^2M(1+M)^2 - LM^2(1+M) - M^3 = 0.$$

Let us introduce the variables

$$a = M - L - LM, \qquad b = (L - M)LM.$$

Then we obtain $Q(1, L, M) = -a^3 + 4b(1+a)$ and the equation defining the variety $V_{\mathbb{C}}$ becomes much simpler:

$$b = \frac{1}{4}\frac{a^3}{1+a}.$$

Next we consider a square root of $a + 1$ and we solve the equations

$$z^2 - 1 = a = M - L - LM, \qquad \frac{1}{4}\frac{(z^2-1)^3}{z^2} = b = (L - M)LM$$

with respect to L and M. Then we obtain four solution pairs $\{L, M\}$ depending on the variable z. For example,

$$L(z) = \frac{-(1+z)(1 - 2z + z^2 + \sqrt{(1+z)(1+3z-5z^2+z^3)})}{4z},$$

$$M(z) = \frac{(1+z)(1 - 2z + z^2 + \sqrt{(1+z)(1+3z-5z^2+z^3)})}{4z}.$$

The polynomial

$$(z+1)(1+3z-5z^2+z^3)=(z+1)(z-1)(z+(2+\sqrt{5}))(z+(2-\sqrt{5}))$$

has four different zeros. The square root $\sqrt{(1+z)(1+3z-5z^2+z^3)}$ is a meromorphic function on the compact Riemann surface of genus $g=1$. Consequently, there exists a torus $\mathbb{C}/\Gamma$ and elliptic functions $L, M : \mathbb{C}/\Gamma \to \mathbb{P}^1(\mathbb{C})$ such that the components of the variety $V_{\mathbb{C}}\backslash V_{\mathbb{C}}^{\mathrm{sing}}$ are parametrized by L and M:

$$V_{\mathbb{C}}=\{[1:L(z):M(z)]:\ \ z\in\mathbb{C}/\Gamma\}.$$

The functions $L-M$ and $L\cdot M$ are given by the formulas:

$$L-M=-\frac{(1+z)(z-1)^2}{2z},\qquad L\cdot M=-\frac{(1+z)^2(z-1)}{2z}.$$

The moduli space we are interested in coincides with the real points of the projective variety $V_{\mathbb{C}}$. If $K=0$, the only solutions of the equation $Q(0,L,M)=0$ are $L=0$ or $M=0$, i.e., the points $[0:1:0]$ and $[0:0:1]$. Therefore we can parametrize the moduli space by the parameter $M\in\mathbb{R}$ solving the equation $Q(1,L,M)=0$ with respect to $L=L(M)$. In this way we obtain a configuration of six curves in $\mathbb{P}^2(\mathbb{R})$ connecting the three singular points of $V_{\mathbb{C}}$ (see the figure in Section 1). However, we obtain geometrically different metrics on S^3 only for two curves parametrized by the real parameter $0\le M\le\infty$. The graphs of the function $L_{\pm}(M)$ are given in Fig. 1.

The functions $L_{\pm}(M)$ are monotone and tend to ±1 in case that M tends to infinity. Let us discuss the geometric invariants of these metrics. The graph of the scalar curvatures $S_{\pm}(M)$ depending on M is given by Fig. 2.

Next we plot the eigenvalues $A_{\pm}(M), B_{\pm}(M), C_{\pm}(M)$ of the Ricci tensor for both families of metrics (Figs. 3 and 4):

In dimension $n=3$ the number

$$\lambda^2(D)\cdot[\mathrm{vol}(N^3)]^{2/3}$$

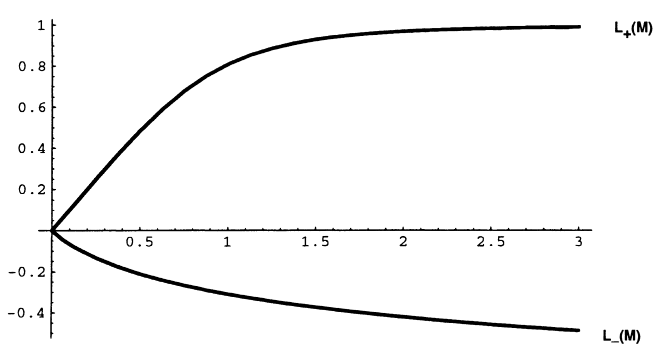


Fig. 1. The graph of $L_{\pm}M$.

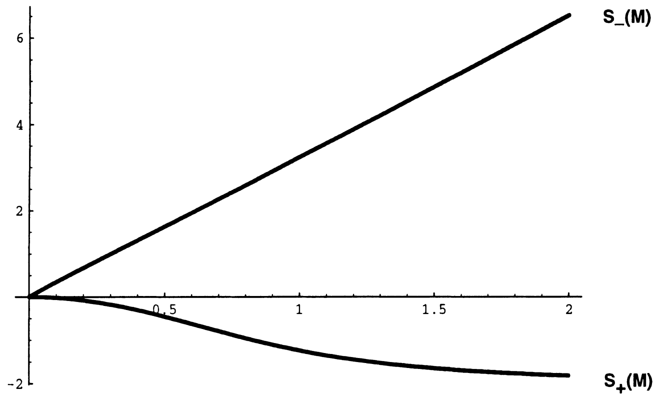


Fig. 2. The scalar curvatures.

is a homothety invariant, where $\lambda(D)$ is an eigenvalue of the Dirac operator. In case of a WK-spinor we have

$$\lambda^2 = \frac{1}{8}\frac{S^3}{S^2 - 2|\mathrm{Ric}|^2}$$

and, therefore, we obtain the formula

$$\lambda^2 \cdot \mathrm{vol}^{2/3} = \frac{1}{8}(2\pi^2)^{2/3}\frac{S^3}{S^2 - 2|\mathrm{Ric}|^2}\frac{1}{\{|K-L||M-L||K+M|\}^{2/3}}.$$

Figs. 5 and 6 contain the graph of $\lambda^2\mathrm{vol}^{2/3}(M)$ depending on the parameter M for both families of metrics.

Finally, let us discuss the behaviour of the rational function

$$\Psi = \frac{L^2}{KM}$$

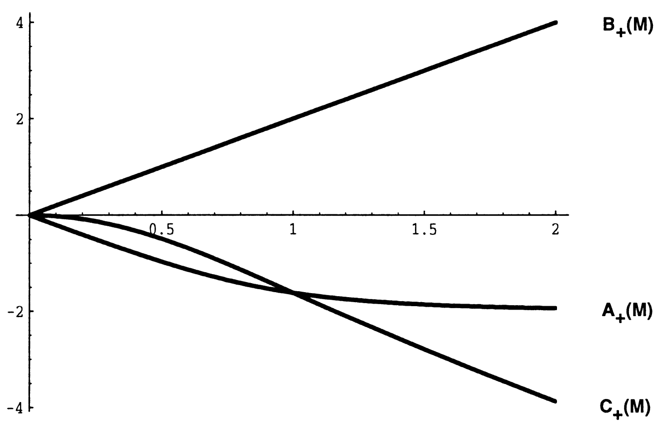


Fig. 3. The eigenvalues of the Ricci tensor for $L_+(M)$.

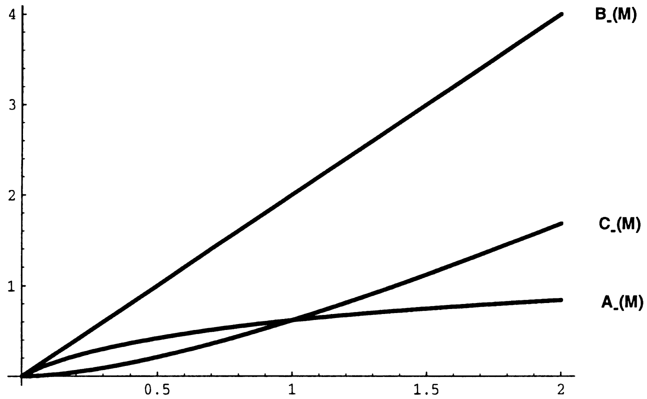


Fig. 4. The eigenvalues of the Ricci tensor for $L_{-}(M)$.

on the variety $V_{\mathbb{C}} \subset \mathbb{P}^2(\mathbb{C})$. It turns out that Ψ has simple zeros at the singular points $[1 : 0 : 0]$ and $[0 : 0 : 1]$. Indeed, solving the equation defining $V_{\mathbb{C}}$ with respect to $L = L(M)(K = 1)$ we obtain

$$\lim_{M\to 0} \frac{L^2(M)}{M} = 0, \qquad \lim_{M\to 0} \frac{\mathrm{d}}{\mathrm{d}M}\left(\frac{L^2(M)}{M}\right) = 1.$$

The third singular point $[0 : 1 : 0]$ is a pole of order two. In the regular part of $V_{\mathbb{C}}$ the function Ψ has 12 ramification points. Among them 10 points are first order ramification points. The ramification points of order two are the points

$$[K : L : M] = [1 : \tfrac{1}{4}(1 \pm \sqrt{5}) : 1].$$

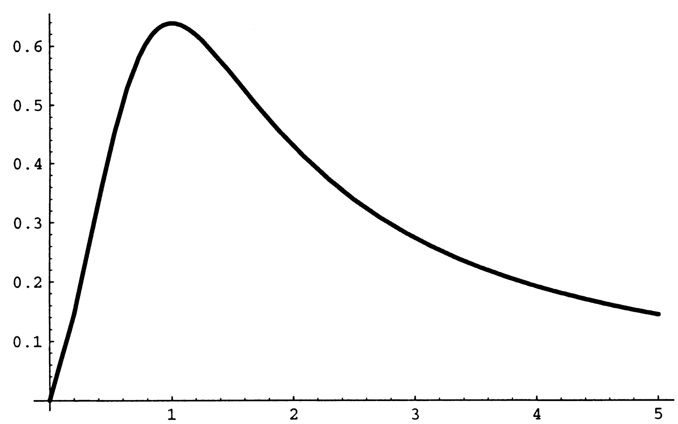


Fig. 5. $\lambda^2 \mathrm{vol}^{2/3}$ in case of $L_{+}(M)$.

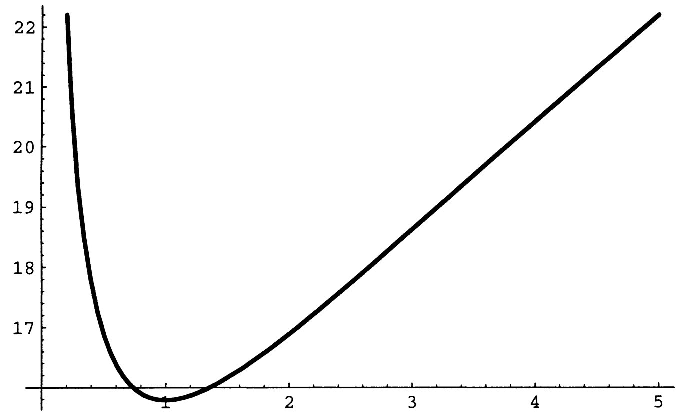


Fig. 6. $\lambda^2\mathrm{vol}^{2/3}$ in case of $L_-(M)$.

These parameters correspond precisely to the non-Einstein–Sasakian metrics on S^3 admitting solutions of the Einstein–Dirac equation.

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